ECEn 462

Transient Analysis of Dielectric Slabs using Finite Differences

Purpose:
The purpose of this laboratory exercise is to understand and use finite-difference approximations in the time-domain for electromagnetic analysis.

Preliminary:
Consider Maxwell's equations in the time-domain, namely

$$\nabla \times \overline{E}(\overline{r}, t) = -\mu \frac{\partial \overline{H}}{\partial t}(\overline{r}, t) \tag{1}$$

$$\nabla \times \overline{H}(\overline{r}, t) = \epsilon \frac{\partial \overline{E}}{\partial t}(\overline{r}, t) \tag{2}$$

1. Show that in 1 dimension in a charge-free environment, the scalar wave equation derived from (1) and (2) may be expressed as

$$\frac{\partial^2 E}{\partial t^2}(z, t) = c^2 \frac{\partial^2 E}{\partial z^2}(z, t) \tag{3}$$

2. Use Taylor series expansions of $E(z, t)$ about the point (z, t) to show that

$$\frac{\partial^2 E}{\partial z^2}(z, t) \approx \frac{E_{i+1}^n - 2E_i^n + E_{i-1}^n}{\Delta z^2} \tag{4}$$

$$\frac{\partial^2 E}{\partial t^2}(z, t) \approx \frac{E_i^{n+1} - 2E_i^n + E_i^{n-1}}{\Delta t^2} \tag{5}$$

   where $E_i^n = E(i\Delta z, n\Delta t)$.

3. Use these difference approximations in (3) and rearrange to show that

$$E_i^{n+1} = \left(\frac{c\Delta t}{\Delta z}\right)^2 \left(E_{i+1}^n - 2E_i^n + E_{i-1}^n\right) + 2E_i^n - E_i^{n-1}. \tag{6}$$

   Describe how you could use this equation to track the time evolution of an electric field traveling through a domain given the initial conditions (at $t = 0$).

   Note: When you implement this in code, notice that at the domain edge (say $i = N$), you must know E_{N+1}^n which is outside the domain for the computation. If you set this value to zero, you will get reflections back into the domain off the end. To work around this problem, implement a more advanced treatment called an absorbing boundary condition. The absorbing boundary conditions can be implemented as:

$$E_{N+1}^{n+1} = E_N^n + \frac{\tau - 1}{\tau + 1}\left[E_N^{n+1} - E_{N+1}^n\right] \tag{7}$$

$$E_0^{n+1} = E_1^n + \frac{\tau - 1}{\tau + 1}\left[E_1^{n+1} - E_0^n\right] \tag{8}$$

where $\tau = c\Delta t/\Delta z$.

4. Describe how to account for dielectric media within the propagation domain in the finite-difference equation.

Procedure:

1. Write a computer program or script (you may use Matlab, Mathematica, Maple, or any other software you have access to) to implement the finite-difference equation. At $z = 0$, enforce a source condition in time such that
$$E(0,t) = e^{-(t-t_o)^2/2\sigma^2}$$
where t_o is chosen such that the pulse is essentially zero at $t = 0$. The parameter σ controls the width of the Gaussian pulse, and therefore should be set reasonably small (say, so that the pulse width is on the order of 10-20 spatial cells wide). Set up your program so that in the center of the domain you have a dielectric slab whose thickness is several times the spatial step size Δz.

2. Run your program and monitor the output for the pulse hitting the dielectric slab. Provide some plots of the reflections and transmissions occurring. **Note: It is important to run the code such that**
$$\frac{c\Delta t}{\Delta z} \leq 1.$$

3. Try to run the code for
$$\frac{c\Delta t}{\Delta z} \gg 1$$
and comment on the results. Provide a physical reason for the results you see.

4. Be creative: try at least one additional (and more complicated) material configuration such as a multilayer slab or multiple slabs at different locations.

Helps:

1. Taylor series expansion:
$$E(z \pm \Delta z, t) = E(z,t) \pm \Delta z \frac{\partial E(z,t)}{\partial z} + \frac{\Delta z^2}{2!}\frac{\partial^2 E(z,t)}{\partial z^2} \pm \frac{\Delta z^3}{3!}\frac{\partial^3 E(z,t)}{\partial z^3} + \cdots \quad (9)$$
$$E(z, t \pm \Delta t) = E(z,t) \pm \Delta t \frac{\partial E(z,t)}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 E(z,t)}{\partial t^2} \pm \frac{\Delta t^3}{3!}\frac{\partial^3 E(z,t)}{\partial t^3} + \cdots \quad (10)$$
Let $z = i\Delta z$ and $t = n\Delta t$. Then try performing
$$E(z+\Delta z, t) + E(z-\Delta z, t) = E_{i+1}^n + E_{i-1}^n \quad (11)$$
$$E(z, t+\Delta t) + E(z, t-\Delta t) = E_i^{n+1} + E_i^{n-1} \quad (12)$$

and solving for $\partial^2 E/\partial z^2$ and $\partial^2 E/\partial t^2$. You will have to neglect higher order terms in the series.

2. When programming, make sure to make t_o large enough (3 to 5 times σ) so that the Gaussian is small at $t = 0$. Also, make σ large enough. If you make $\Delta t = 1$, you can make $\sigma = 10$, $t_o = 4\sigma$ and it should work pretty well.

3. Note that you don't have to explicitly set Δz, but can just set $c\Delta t/\Delta z$. Make sure this is less than unity unless otherwise specified.

4. The slab is characterized by $\epsilon_r < 1$. Since $c = 1/\sqrt{\mu_o \epsilon_o \epsilon_r}$, this should indicate how to put the slab into the computer model. For example, if in free-space you set $c_o \Delta t/\Delta z = 0.95$, in a slab with $\epsilon_r = 4$ you would use $c\Delta t/\Delta z = 0.95/2$

5. To watch the pulse in a "movie" without explicitly creating a Matlab movie, you can use the following. This code also draws vertical lines where the slab is located. Suppose you have set `sb` and `se` to be the beginning and ending pixels of the slab. The entire domain is `N+1` spatial pixels long. Then you can define

```
x1 = [sb sb];
x2 = [se se];
y = [-1 1];
x = 1:N+1;
```

In the loop where you are computing the fields, you can create the plot of the electric field with the slab using

```
plot(x,e(:,3),'-',x1,y,'-',x2,y,'-');
axis([0,N,-1.2,1.2]);
pause(0.1);
```